Exact and approximate solutions to Schrödinger's equation with decatic potentials

David Brandon and Nasser Saad

Department of Mathematics and Statistics, University of Prince Edward Island,
550 University Avenue, Charlottetown, PEI, Canada C1A 4P3.[*]

Abstract: The one-dimensional Schrödinger's equation is analysed with regard to the existence of exact solutions for decatic polynomial potentials. Under certain conditions on the potential's parameters, we show that the decatic polynomial potential $V(x) = ax^{10} + bx^8 + cx^6 + dx^4 + ex^2$, $a > 0$ is exactly solvable. By examining the polynomial solutions of certain linear differential equations with polynomial coefficients, the necessary and sufficient conditions for corresponding energy-dependent polynomial solutions are given in detail. It is also shown that these polynomials satisfy a four-term recurrence relation, whose real roots are the exact energy eigenvalues. Further, it is shown that these polynomials generate the eigenfunction solutions of the corresponding Schrödinger equation. Further analysis for arbitrary values of the potential parameters using the asymptotic iteration method is also presented.

I. INTRODUCTION

The problem of finding analytic (polynomial) solutions of linear differential equations has lost some of its interest in recent years. The lack of interest is mainly due to the vast development of numerical methods and the fact that much of their study has been superseded by more general work based on the theory of Lie algebra [27, 29, 39, 40, 56, 57, 66]. Recently, the interest of analytic (polynomial) solutions has been renewed by the analysis of confined and un-confined quantum systems and also by the search of closed-form solutions to (Schrödinger-type) differential equations with polynomial coefficients [1, 3–5, 7, 10–13, 16–23, 26, 32–34, 43, 46–48, 54–56, 60, 63–65, 67]. A reason for such interest is that in many problems in quantum mechanics, especially those arising from Schrödinger's equation after the separation of the asymptotic-behaviour factor of the wave function, there remains a polynomial-type factor in the solution [2–7, 11, 16, 17, 22, 23, 26, 27, 29, 31–35, 39, 40, 42, 46, 51–53, 55–57, 61, 67]. One can further argue that analytic (polynomial) solutions provide a deeper quantitative insight into the physical model under investigation and in many cases makes the conceptual understanding of physics straightforward and sometimes intuitive [6, 24, 25, 45, 59]. Moreover, these solutions (if they are available) are valuable tools for checking and improving numerical methods introduced for solving complicated physical systems [6, 54].

In the present work we study the exact and approximate solutions of the Schrödinger equation with decatic polynomial potential [3, 4, 19, 22, 49]

$$\hat{H}\psi = -\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi, \quad V(x) = ax^{10} + bx^8 + cx^6 + dx^4 + ex^2 \qquad (a > 0, \ -\infty < x < \infty). \tag{1}$$

The importance of this type of even-power potentials follows from its relevance to different models of charmonium system [25, 59] and its connection with the analysis of anharmonic and double-well oscillators [9, 15, 21, 24, 25, 30, 37, 44, 45, 49, 51, 59, 62]. We show under certain relations between the potential parameters a, b, c, d, and e, we have exact solutions for the Schrödinger equation (1), in the sense that ψ and E are completely determined. These relations are expressed in terms of energy-dependent polynomials that satisfies a four-term recurrence relation. We show further that these polynomials generate the corresponding eigenfunctions. Our approach depends on analysing the analytic (polynomial-type) solutions of a second-order differential equation

$$\left(\sum_{k=0}^{6} a_{6,k} x^{6-k}\right) y'' + \left(\sum_{k=0}^{5} a_{5,k} x^{5-k}\right) y' - \left(\sum_{k=0}^{4} \tau_{4,k} x^{4-k}\right) y = 0, \qquad (a_{6,0}^2 + a_{5,0}^2 \neq 0), \tag{2}$$

---

[*]Electronic address: dbrandon@upei.ca; Electronic address: nsaad@upei.ca

where $a_{6,k}, a_{5,k}$ and $\tau_{4,k}$, for all k, are constants independent of the variable x. Thereby, we give the necessary and sufficient conditions for its polynomial solutions $y = \sum_{k=0}^{n} c_k x^k$. In the second part of the present study, we introduce the asymptotic iteration method [16] to obtain accurate approximate solutions for the decatic potential (1) with arbitrary values of the potential parameters. In section II, we set up the Schrödinger equation for the decatic-power potential (1) and we show it reduces to a differential equation of a type similar to equation (2). In section III, we provides both the necessary and sufficient conditions for the existence of polynomial solutions of the (general) linear differential equation (2). In section IV, we examine the exact and approximate solutions of decatic potential (2) and report some numerical results illustrate the usefulness of our approach where we compare our results with the existing literature.

II.   ONE-DIMENSIONAL DECATIC-POWER POTENTIAL

In this section, we consider the one-dimensional Schrödinger equation

$$-\psi''(x) + (V(x) - E)\psi(x) = 0, \qquad V(x) = ax^{10} + bx^8 + cx^6 + dx^4 + ex^2, \quad -\infty < x < \infty, \tag{3}$$

where a, b, c, d, and e are real constants with $a > 0$. Assume the solution has the form

$$\psi(x) = \chi(x) \cdot e^{-\phi(x)} \qquad \left(\lim_{|x| \to \infty} \phi(x) = \infty \right). \tag{4}$$

Substituting this expression into equation (3) gives a second-order linear differential equation for $\chi(x)$ as

$$\chi''(x) = 2\phi'(x)\chi'(x) + \left(\phi''(x) - \left(\phi'(x)\right)^2 + V(x) - E \right)\chi(x). \tag{5}$$

Without loss of generality, we may assume, for the nodeless eigenstate $\psi_0(x)$, that $\chi(x) = 1$. In which case, equation (5) reduces to Riccati's equation

$$\phi''(x) = E - V(x) + \left(\phi'(x)\right)^2 \quad \text{or} \quad u'(x) = E - V(x) + u^2(x), \quad \text{where} \quad u(x) = \phi'(x). \tag{6}$$

Definition 1 *([50], page 474) By the symbol $\left[\sqrt{P(x)} \right]$, where $P(x)$ is a polynomial of even-degree, we shall mean the polynomial part of the expansion of $\sqrt{P(x)}$ in a series of descending integral powers of x.*

Theorem 1 *([50], Theorem 1, page 474) If in $du/dx = A_0 + u^2$, $A_0 \equiv A_0(x)$ is a polynomial of even degree, then no polynomial other than*

$$u = \pm \left[\sqrt{-A_0} \right] \tag{7}$$

can be a solution of (6). If the degree of A_0 is odd, there is no polynomial solution of (6).

By means of this theorem, we obtain for the solution of equation (3)

$$\psi(x) = \chi(x) \cdot \exp\left(-\frac{\sqrt{a}}{6}x^6 - \frac{b}{8\sqrt{a}}x^4 + \frac{(b^2 - 4ac)}{16a^{3/2}}x^2 \right), \tag{8}$$

where $\chi(x)$ satisfy the second-order linear differential equation

$$\chi''(x) - \left(2\sqrt{a}x^5 + \frac{b}{\sqrt{a}}x^3 - \frac{(b^2 - 4ac)}{4a^{3/2}}x \right)\chi'(x)$$
$$- \left(\frac{4ac - b^2}{8a^{3/2}} - E + \frac{64a^3 e + 96a^{5/2}b - b^4 + 8ab^2 c - 16a^2 c^2}{64a^3}x^2 + \frac{8a^2 d + 40a^{5/2} + b^3 - 4abc}{8a^2}x^4 \right)\chi(x) = 0. \tag{9}$$

III. NECESSARY AND SUFFICIENT CONDITIONS FOR POLYNOMIAL SOLUTIONS

In this section, we study a class of differential equations that generalized equation (9) and we establish the necessary and sufficient conditions for the existence of polynomial solutions.

Theorem 2 *The necessary condition for the existence of polynomial solutions of the differential equation*

$$(a_{6,0}x^6 + a_{6,1}x^5 + a_{6,2}x^4 + a_{6,3}x^3 + a_{6,4}x^2 + a_{6,5}x + a_{6,6})y''$$
$$+(a_{5,0}x^5 + a_{5,1}x^4 + a_{5,2}x^3 + a_{5,3}x^2 + a_{5,4}x + a_{5,5})y' - (\tau_{4,0}x^4 + \tau_{4,1}x^3 + \tau_{4,2}x^2 + \tau_{4,3}x + \tau_{4,4})y = 0, \quad (10)$$

where $a_{6,0}^2 + a_{5,0}^2 \neq 0$ and at least one of the coefficients $a_{6,j}, j = 0, 1, \ldots, 6$ is different from zero, is

$$\tau_{4,0} = n(n-1) \cdot a_{6,0} + n \cdot a_{5,0} \qquad (n = 0, 1, 2, \ldots) \tag{11}$$

while the sufficient condition is given by the solution of the following system of linear equations

$$
\begin{pmatrix}
\alpha_0 & \beta_0 & \gamma_0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
\delta_1 & \alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
\eta_2 & \delta_2 & \alpha_2 & \beta_2 & \gamma_2 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
\mu_3 & \eta_3 & \delta_3 & \alpha_3 & \beta_3 & \gamma_3 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
\zeta_4 & \mu_4 & \eta_4 & \delta_4 & \alpha_4 & \beta_4 & \gamma_4 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & \zeta_5 & \mu_5 & \eta_5 & \delta_5 & \alpha_5 & \beta_5 & \gamma_5 & \ldots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \eta_{n-2} & \delta_{n-2} & \alpha_{n-2} & \beta_{n-2} & \gamma_{n-2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \mu_{n-1} & \eta_{n-1} & \delta_{n-1} & \alpha_{n-1} & \beta_{n-1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & \zeta_n & \mu_n & \eta_n & \delta_n & \alpha_n
\end{pmatrix}_{(n+1)\times(n+1)}
\cdot
\begin{pmatrix}
c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ \vdots \\ c_{n-2} \\ c_{n-1} \\ c_n
\end{pmatrix}_{(n+1)\times 1}
= 0 \tag{12}
$$

where

$$
\begin{cases}
\alpha_n = n(n-1)a_{6,4} + na_{5,4} - \tau_{4,4}, & n = 0, 1, 2, \ldots \\
\beta_n = n(n+1)a_{6,5} + (n+1)a_{5,5}, & n = 0, 1, 2, \ldots \\
\gamma_n = (n+2)(n+1)a_{6,6}, & n = 0, 1, 2, \ldots \\
\delta_n = (n-1)(n-2)a_{6,3} + (n-1)a_{5,3} - \tau_{4,3}, & n = 1, 2, \ldots \\
\eta_n = (n-2)(n-3)a_{6,2} + (n-2)a_{5,2} - \tau_{4,2}, & n = 2, 3, \ldots \\
\mu_n = (n-3)(n-4)a_{6,1} + (n-3)a_{5,1} - \tau_{4,1}, & n = 3, 4, \ldots \\
\zeta_n = (n-4)(n-5)a_{6,0} + (n-4)a_{5,0} - \tau_{4,0}, & n = 4, 5, \ldots
\end{cases}
\tag{13}
$$

In particular, for a zero-degree polynomial solution of the differential equation (10) the sufficient and necessary conditions are

$$\tau_{4,0} = \tau_{4,1} = \tau_{4,2} = \tau_{4,3} = \tau_{4,4} = 0, \tag{14}$$

while for a first-degree polynomial solution of equation (10), we must have $\tau_{4,0} = a_{5,0}$ in addition to vanishing the four 2×2-determinants

$$
\begin{vmatrix} -\tau_{4,4} & a_{5,5} \\ -\tau_{4,3} & a_{5,4} - \tau_{4,4} \end{vmatrix} = 0, \quad
\begin{vmatrix} -\tau_{4,4} & a_{5,5} \\ -\tau_{4,2} & a_{5,3} - \tau_{4,3} \end{vmatrix} = 0, \quad
\begin{vmatrix} -\tau_{4,4} & a_{5,5} \\ -\tau_{4,1} & a_{5,2} - \tau_{4,2} \end{vmatrix} = 0, \quad
\begin{vmatrix} -\tau_{4,4} & a_{5,5} \\ -\tau_{4,0} & a_{5,1} - \tau_{4,1} \end{vmatrix} = 0. \tag{15}
$$

For a second-degree polynomial solution, we must have $\tau_{4,0} = 2 \cdot a_{6,0} + 2 \cdot a_{5,0}$ in addition to the vanishing of the four 3×3-determinants

$$
\begin{vmatrix}
-\tau_{4,4} & a_{5,5} & 2a_{6,6} \\
-\tau_{4,3} & a_{5,4} - \tau_{4,4} & 2a_{6,5} + 2a_{5,5} \\
-\tau_{4,2} & a_{5,3} - \tau_{4,3} & 2a_{6,4} + 2a_{5,4} - \tau_{4,4}
\end{vmatrix} = 0, \quad
\begin{vmatrix}
-\tau_{4,4} & a_{5,5} & 2a_{6,6} \\
-\tau_{4,3} & a_{5,4} - \tau_{4,4} & 2a_{6,5} + 2a_{5,5} \\
-\tau_{4,1} & a_{5,2} - \tau_{4,2} & 2a_{6,3} + 2a_{5,3} - \tau_{4,3}
\end{vmatrix} = 0,
$$

$$
\begin{vmatrix}
-\tau_{4,4} & a_{5,5} & 2a_{6,6} \\
-\tau_{4,3} & a_{5,4} - \tau_{4,4} & 2a_{6,5} + 2a_{5,5} \\
-\tau_{4,0} & a_{5,1} - \tau_{4,1} & 2a_{6,2} + 2a_{5,2} - \tau_{4,2}
\end{vmatrix} = 0, \quad
\begin{vmatrix}
-\tau_{4,4} & a_{5,5} & 2a_{6,6} \\
-\tau_{4,3} & a_{5,4} - \tau_{4,4} & 2a_{6,5} + 2a_{5,5} \\
0 & a_{5,0} - \tau_{4,0} & 2a_{6,1} + 2a_{5,1} - \tau_{4,1}
\end{vmatrix} = 0. \tag{16}
$$

Similar conditions follows for higher order of polynomial solutions. The first few polynomial solutions are given explicitly by

$$y_0(x) = 1, \qquad y_1(x) = 1 + \frac{\tau_{4,4}}{a_{5,5}} \cdot x,$$

$$y_2(x) = 1 + \frac{-a_{6,6}\tau_{4,3} + (a_{5,5} + a_{6,5})\tau_{4,4}}{a_{5,5}(a_{5,5} + a_{6,5}) + a_{6,6}(-a_{5,4} + \tau_{4,4})} \cdot x + \frac{1}{2} \cdot \frac{a_{5,5}\tau_{4,3} - a_{5,4}\tau_{4,4} + \tau_{4,4}^2}{(a_{5,5}(a_{5,5} + a_{6,5}) + a_{6,6}(-a_{5,4} + \tau_{4,4}))} \cdot x^2. \qquad (17)$$

This theorem can be proved either by means of the Frobenius series solution of ordinary Differential Equations or using the asymptotic iteration method [16, 17, 53].

IV. THE ONE-DIMENSIONAL DECATIC POLYNOMIAL POTENTIALS

For the existence of polynomial solutions of the differential equation (9), it is clear from the criterion (11) that the parameters a, b, c, and d must be related through the equation

$$8(5 + 2n)a^{\frac{5}{2}} + 8a^2d - 4abc + b^3 = 0, \qquad n = 0, 1, 2, \ldots \qquad (18)$$

There is an important consequence of this criterion that *none* of the following decatic polynomials

$$V(x) = a \cdot x^{10}, \quad V(x) = a \cdot x^{10} + e \cdot x^2, \quad V(x) = a \cdot x^{10} + c \cdot x^6, \quad V(x) = a \cdot x^{10} + c \cdot x^6 + e \cdot x^2, \qquad (19)$$

have analytic solutions of the form (8) with $\chi(x)$ is a polynomial-type expression. Indeed, we can claim out of this criterion that none of these potentials (19) is a quasi-exactly solvable model [58] and we have to rely on perturbation or approximation methods to obtain the necessary information about the discrete spectrum of these potentials as is usually used in the literature (e.g. [8, 9, 15, 24, 28, 36, 38, 41, 42, 62]). In the following, we shall focus on studying the decatic potentials (3) using Theorem 2 to obtain the necessary conditions on the potential parameters for exact solutions followed by the application of the asymptotic iteration method to analyse the discrete spectrum for arbitrary values of the potential parameters.

A. Exact solutions

For the necessary condition for even-degree polynomial solutions of Schrödinger equation (3) we have, by means of equation (11) that

$$8(4n + 5)a^{\frac{5}{2}} + 8a^2d - 4abc + b^3 = 0, \qquad n = 0, 1, 2, \ldots \qquad (20)$$

while for the sufficient condition we deduce, by means of equations (12)-(13), that

$$\chi_{2n+2}(x) = \sum_{k=0}^{n} \frac{(-1)^k \cdot P_{2k}(E)}{(2k)! \cdot (2\sqrt{a})^{3k}} \cdot x^{2k}, \qquad (n = 0, 1, 2, \ldots). \qquad (21)$$

where the polynomials $P_{2k}(E)$ satisfies the four-term recurrence relation

$$P_{2n+2}(E) \equiv \left(8a^{3/2}E + (4n + 1)\left(b^2 - 4ac\right) \right) \cdot P_{2n}(E)$$
$$+ 2n \cdot (2n - 1) \cdot (32(4n - 1)a^{5/2}b - b^4 + 8ab^2c - 16a^2c^2 + 64a^3e) \cdot P_{2n-2}(E)$$
$$+ 16384 \cdot a^5 \cdot n \cdot (n - 1) \cdot (2n - 1) \cdot (2n - 3) \cdot P_{2n-4}(E), \qquad (22)$$

initiated with $P_0(E) = 1, P_{-n}(E) = 0, n \geq 1$. The exact (real) eigenenergies E are the roots of equation $P_{2n+2}(E) = 0$ subject to the following additional condition

$$\left(32(4n+3)a^{5/2}b - b^4 + 8ab^2c - 16a^2c^2 + 64a^3e \right) \cdot P_{2n}(E) + 4096 \cdot a^5 \cdot n \cdot (2n-1) \cdot P_{2n-2}(E) = 0, \qquad (n = 0, 1, 2, \ldots). \quad (23)$$

As an example, for the ground state eigenvalue $n = 0$, we have for arbitrary values of the potential parameters a, b, and c that

$$E_0 = \frac{1}{8} \cdot \frac{4ac - b^2}{a^{3/2}}, \quad d = -\frac{1}{8} \cdot \frac{40a^{5/2} + b^3 - 4abc}{a^2}, \quad e = -\frac{1}{64} \cdot \frac{96a^{5/2}b - b^4 + 8ab^2c - 16a^2c^2}{a^3} \qquad (24)$$

On the other hand, the necessary condition for odd-degree polynomial solutions of Schrödinger equation (3), by means of equation (11), is

$$8(4n+7)a^{\frac{5}{2}} + 8a^2 d - 4abc + b^3 = 0, \qquad (n = 0, 1, 2, \dots) \tag{25}$$

under which the exact wavefunctions are given by

$$\chi_{2n+3}(x) = \sum_{k=0}^{n} \frac{(-1)^k \cdot P_{2k+1}(E)}{(2k+1)! \cdot (2\sqrt{a})^{3k}} x^{2k+1}, \qquad n = 0, 1, 2, \dots \tag{26}$$

where the polynomials $P_{2k+1}(E)$ are given explicitly using the recurrence relation

$$P_{2n+3}(E) \equiv \left(8a^{3/2} E + (4n+3)\left(b^2 - 4ac \right) \right) \cdot P_{2n+1}(E)$$

$$+ 2n \cdot (2n+1) \cdot \left(32(4n+1)a^{5/2}b - b^4 + 8ab^2 c - 16a^2 c^2 + 64a^3 e \right) \cdot P_{2n-1}(E)$$

$$+ 16384 \cdot a^5 \cdot n \cdot (n-1) \cdot (2n-1) \cdot (2n+1) \cdot P_{2n-3}(E), \qquad (n = 0, 1, 2, \dots) \tag{27}$$

initiated with $P_1(E) = 1$ and $P_{-n} = 0$ for $n \geq 1$. The exact eigenenergies E are the real roots of $P_{2n+3}(E) = 0$, subject to the additional condition

$$\left(32(4n+5)a^{5/2}b - b^4 + 8ab^2 c - 16a^2 c^2 + 64a^3 e \right) \cdot P_{2n+1}(E) + 4096 \cdot a^5 \cdot n \cdot (2n+1) \cdot P_{2n-1}(E) = 0, \qquad (n = 0, 1, 2, \dots). \tag{28}$$

As an example, for the first excited state $n = 1$, we have for the potential parameters a, b, and c that

$$E_1 = \frac{3}{8} \cdot \frac{4ac - b^2}{a^{3/2}}, \quad d = -\frac{1}{8} \cdot \frac{56a^{5/2} + b^3 - 4abc}{a^2}, \quad e = -\frac{1}{64} \cdot \frac{160a^{5/2}b - b^4 + 8ab^2 c - 16a^2 c^2}{a^3} \tag{29}$$

The first few polynomial solutions, $n = 0, 1, \dots, 4$, are reported explicitly in the appendix. In Table I, we report the ground-state eigenvalue for the decatic potentials (3) for specific values of the potential parameters a, b, c, d and e. Further, in Table II, we report the first-excited state for particular values of the potential parameters. These results confirm and generalize the early finding by Chaudhuri and Mondal [15], using the improved Hill determinant method, for the particular potentials:

$$V(x) = \frac{105}{64}x^2 - \frac{43}{8}x^4 + x^6 - x^8 + x^{10}, \qquad \left(E_0 = \frac{3}{8}, \ \psi_0(x) = \exp\left(-\frac{3}{16} \cdot x^2 + \frac{1}{8} \cdot x^4 - \frac{1}{6} \cdot x^6 \right) \right), \tag{30}$$

$$V(x) = \frac{169}{64}x^2 - \frac{59}{8}x^4 + x^6 - x^8 + x^{10}, \qquad \left(E_1 = \frac{9}{8}, \ \psi_1(x) = x \cdot \exp\left(-\frac{3}{16} \cdot x^2 + \frac{1}{8} \cdot x^4 - \frac{1}{6} \cdot x^6 \right) \right). \tag{31}$$

The ground-state $n = 0$ and first-excited state $n = 1$ along with the un-normalized wave functions for the potentials (30) and (31) are displayed in Figure 1.
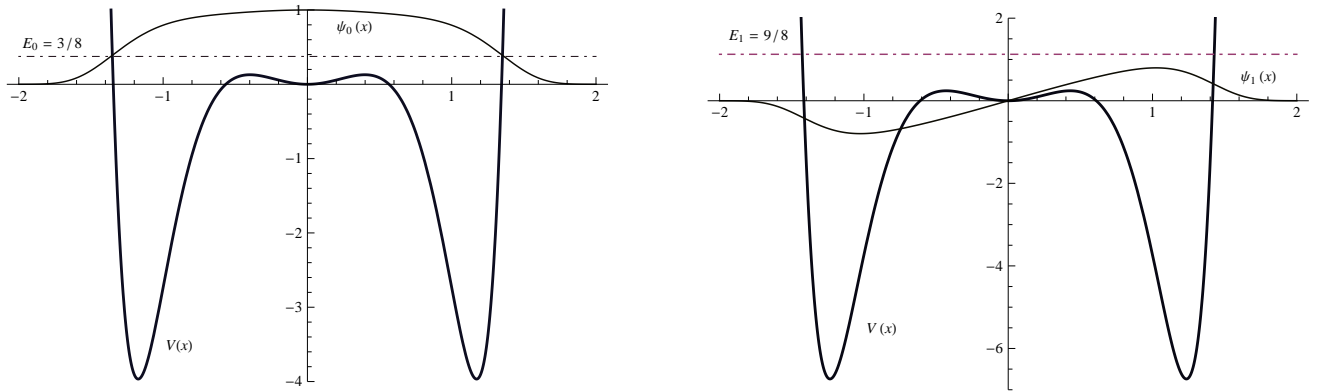

FIG. 1: The ground-state $n = 0$ and first-excited state $n = 1$ (un-normalized) wave functions for the decatic potential energy $V(x) = x^{10} + x^8 + x^6 + dx^4 + ex^2$ along with the exact eigenvalues as are given by (30) and (31), respectively.

E_0	a	b	c	d	e
$3\sqrt{\mu}/8$	μ	$\pm\mu$	μ	$\frac{1}{8}\cdot(\pm3\mu-40\sqrt{\mu})$	$\frac{3}{64}\cdot(3\mu\mp32\sqrt{\mu})$
$-5\sqrt{\mu}/8$	μ	$\pm\mu$	$-\mu$	$\mp\frac{5}{8}\cdot(\mu\pm8\sqrt{\mu})$	$\frac{1}{64}\cdot(25\mu\mp96\sqrt{\mu})$
$(4k-1)\sqrt{k\mu}/(8k^2)$	$k\cdot\mu$	μ	μ	$-\frac{1}{8k^2}\cdot(40k^2\sqrt{k\mu}+(1-4k)\mu)$	$\frac{1}{64k^3}\cdot((1-4k)^2\mu-96k^2\sqrt{k\mu})$
$-(k^2-4)\sqrt{\mu}/8$	μ	$k\cdot\mu$	μ	$-\frac{1}{8}\cdot k\cdot(k^2-4)\cdot\mu-5\cdot\sqrt{\mu}$	$\frac{1}{64}\cdot(k^2-4)^2\cdot\mu-\frac{3}{2}\cdot k\cdot\sqrt{\mu}$
$(4k-1)\sqrt{\mu}/8$	μ	μ	$k\cdot\mu$	$\frac{1}{8}\cdot(4k-1)\cdot\mu-5\cdot\sqrt{\mu}$	$\frac{1}{64}\cdot(1-4k)^2\mu-\frac{3}{2}\cdot\sqrt{\mu}$
$\sqrt{k\cdot\mu}+5\cdot k$	$k\cdot\mu$	μ	$\frac{\mu}{4k}+2k\cdot(\mu+5\sqrt{\mu})$	μ	$25k^2-\frac{3}{2}\sqrt{\frac{\mu}{k}}+k(\mu+10\sqrt{k\mu})$
$(\sqrt{\mu}+5)/k$	μ	$k\cdot\mu$	$\frac{\sqrt{\mu}}{4k}\cdot(40+\sqrt{\mu}\cdot(8+k^3))$	μ	$\frac{1}{2k^2}(2\mu+(20-3k^3)\sqrt{\mu}+50)$
$5+k\cdot\sqrt{\mu}$	μ	μ	$\frac{\sqrt{\mu}}{4}\cdot(40+\sqrt{\mu}\cdot(1+8\cdot k))$	$k\cdot\mu$	$k^2\cdot\mu+(10\cdot k-\frac{3}{2})\cdot\sqrt{\mu}+25$

TABLE I: The ground-state energy E_0 for a decatic-power potential $V(x)=a\cdot x^{10}+b\cdot x^8+c\cdot x^6+d\cdot x^4+e\cdot x^2$ for different values of the parameters. Here, μ and k are arbitrary positive numbers.

E_1	a	b	c	d	e
$9\sqrt{\mu}/8$	μ	$\pm\mu$	μ	$\frac{1}{8}\cdot(\pm3\mu-56\sqrt{\mu})$	$\frac{1}{64}\cdot(9\mu\mp160\sqrt{\mu})$
$-15\sqrt{\mu}/8$	μ	$\pm\mu$	$-\mu$	$\frac{1}{8}\cdot(\mp5\mu-56\sqrt{\mu})$	$\frac{5}{64}\cdot(5\mu\mp32\sqrt{\mu})$
$3(4k-1)\sqrt{k\cdot\mu}/(8k^2)$	$k\cdot\mu$	μ	μ	$-\frac{1}{8k^2}\cdot(56k^2\sqrt{k\mu}+\mu-4k\mu)$	$\frac{1}{64k^3}(-160k^2\sqrt{k\mu}+(1-4k)^2\mu)$
$-\frac{3}{8}(k^2-4)\sqrt{\mu}$	μ	$k\cdot\mu$	μ	$-\frac{k}{8}(k^2-4)\mu-7\cdot\sqrt{\mu}$	$\frac{1}{64}(1-4k)^2\mu-\frac{5}{2}\cdot k\cdot\sqrt{\mu}$
$\frac{3}{8}(4k-1)\sqrt{\mu}$	μ	μ	$k\cdot\mu$	$-\frac{1}{8}\cdot\mu+\frac{1}{2}\cdot k\cdot\mu-7\cdot\sqrt{\mu}$	$\frac{1}{64}\cdot\mu-\frac{1}{8}\cdot k\cdot\mu+\frac{1}{4}\cdot k^2\cdot\mu-\frac{5}{2}\cdot\sqrt{\mu}$
$3\sqrt{k\cdot\mu}+21\cdot k$	$k\cdot\mu$	μ	$\frac{1}{4}\cdot\frac{56\cdot(k\cdot\mu)^{5/2}+\mu^3\cdot(1+8\cdot k^2)}{k\cdot\mu^2}$	μ	$\frac{1}{2}\cdot\frac{-5\cdot\sqrt{k\cdot\mu}+98k^3+28\cdot k^2\cdot\sqrt{k\cdot\mu}+2\cdot k^2\cdot\mu}{k}$
$3(\sqrt{\mu}+7)/k$	μ	$k\cdot\mu$	$\frac{\sqrt{\mu}}{4k}\cdot(56+\sqrt{\mu}\cdot(k^3+8))$	μ	$-\frac{1}{2k^2}\cdot(98+(28-5k^3)\sqrt{\mu}+2\mu)$
$3(7+k\cdot\sqrt{\mu})$	μ	μ	$\frac{1}{4}\cdot(56\cdot\sqrt{\mu}+\mu\cdot(1+8\cdot k))$	$k\cdot\mu$	$k^2\cdot\mu+14\cdot k\cdot\sqrt{\mu}-\frac{5}{2}\cdot\sqrt{\mu}+49$

TABLE II: The first excited state energy E_1 for a decatic-power potential. Here, μ and k are arbitrary positive numbers.

For higher states, we note, using the results reported in the appendix or using Theorem 2 directly, for the potential

$$V(x)=x^{10}+x^8+x^6-\frac{69}{8}x^4$$
$$+\frac{24\left(1971-24\sqrt{6423}\right)^{\frac{1}{3}}-8(1971-24\sqrt{6423})^{\frac{2}{3}}+24(1971+24\sqrt{6423})^{\frac{1}{3}}-8(1971+24\sqrt{6423})^{\frac{2}{3}}-1551}{576}x^2, \tag{32}$$

the exact eigenvalue and wave function are given, in the case $n=2$, by

$$E_2=\frac{1}{24}\left(21+4(1971-24\sqrt{6423}))^{\frac{1}{3}}+4(1971+24\sqrt{6423}))^{\frac{1}{3}}\right) \tag{33}$$

$$\psi_2(x)=\left(1-\frac{1}{12}\left(21+4(1971-24\sqrt{6423}))^{\frac{1}{3}}+4(1971+24\sqrt{6423}))^{\frac{1}{3}}\right)\cdot x^2\right)\cdot\exp\left(-\frac{1}{6}\cdot x^6-\frac{1}{8}\cdot x^4-\frac{3}{16}\cdot x^2\right). \tag{34}$$

Further, for the potential

$$V(x)=x^{10}-x^8+x^6-\frac{75}{8}\cdot x^4$$
$$+\frac{64\sqrt[3]{3}(43+\sqrt{4215})-64\sqrt[6]{3}(23\sqrt{3}+\sqrt{1405})(513+8\sqrt{4215})^{\frac{1}{3}}+571(513+8\sqrt{4215})^{\frac{2}{3}}}{192(513+8\sqrt{4215})^{\frac{2}{3}}}\cdot x^2, \tag{35}$$

we have the exact solutions

$$E_2=\frac{1}{8}\left(7+\frac{4}{3^{\frac{2}{3}}}\cdot(513+8\sqrt{4215})^{\frac{1}{3}}-\frac{52}{3(513+8\sqrt{4215}))^{\frac{1}{3}}}\right), \tag{36}$$

$$\psi_2(x)=\left(1-\frac{1}{16}\left(4+\frac{4}{3^{\frac{2}{3}}}\cdot(513+8\sqrt{4215})^{\frac{1}{3}}-\frac{52}{3(513+8\sqrt{4215}))^{\frac{1}{3}}}\right)\cdot x^2\right)\cdot\exp\left(-\frac{1}{6}\cdot x^6-\frac{1}{8}\cdot x^4-\frac{3}{16}\cdot x^2\right). \tag{37}$$

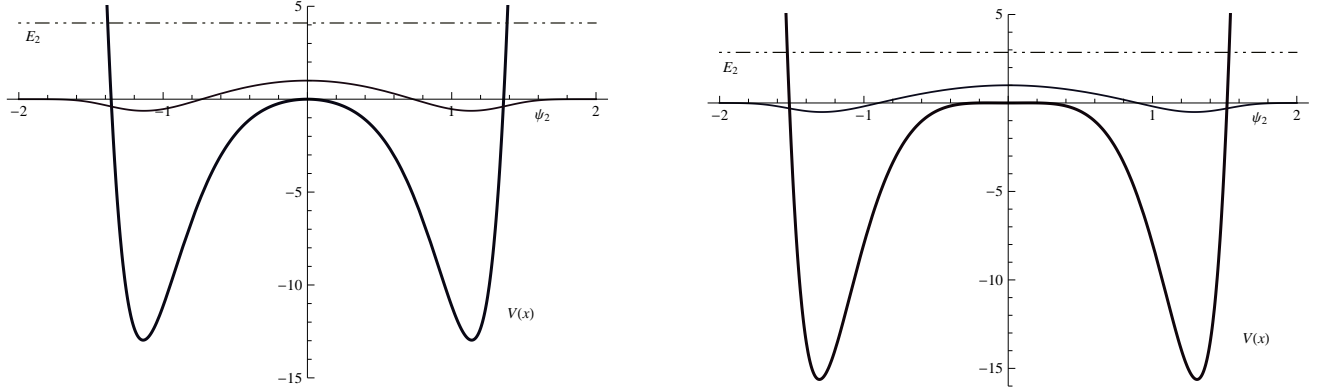The plots of these potentials along with their exact solutions for $n = 2$ are displayed in figure 2.

FIG. 2: The decatic potentials (32) and (35), respectively, and their exact solutions.

B. Approximate solutions

The asymptotic iteration method was introduced [16] to investigate the solutions of differential equations of the form

$$y'' = \lambda_0(x)y' + s_0(x)y, \qquad (' = \frac{d}{dx}) \tag{38}$$

where $\lambda_0 \equiv \lambda_0(x)$ and $s_0 \equiv s_0(x)$ are $C^\infty-$differentiable functions. A key feature of this method is to note the invariant structure of the right-hand side of (38) under further differentiation. Indeed, if we differentiate (38) with respect to x, we obtain

$$y''' = \lambda_1 \cdot y' + s_1 \cdot y \tag{39}$$

where $\lambda_1 = \lambda_0' + s_0 + \lambda_0^2$ and $s_1 = s_0' + s_0\lambda_0$. If we find the second derivative of equation (38), we obtain

$$y^{(4)} = \lambda_2 \cdot y' + s_2 \cdot y \tag{40}$$

where $\lambda_2 = \lambda_1' + s_1 + \lambda_0\lambda_1$ and $s_2 = s_1' + s_0\lambda_1$. Thus, for $(n+1)^{th}$ and $(n+2)^{th}$ derivative of (38), $n = 1, 2, \ldots$, we have

$$y^{(n+1)} = \lambda_{n-1} \cdot y' + s_{n-1} \cdot y \tag{41}$$

and

$$y^{(n+2)} = \lambda_n \cdot y' + s_n \cdot y \tag{42}$$

respectively, where

$$\lambda_n = \lambda_{n-1}' + s_{n-1} + \lambda_0\lambda_{n-1} \quad \text{and} \quad s_n = s_{n-1}' + s_0\lambda_{n-1}. \tag{43}$$

From (41) and (42) we have

$$\lambda_n y^{(n+1)} - \lambda_{n-1}y^{(n+2)} = \delta_n y \quad \text{where} \quad \delta_n = \lambda_n s_{n-1} - \lambda_{n-1}s_n. \tag{44}$$

Clearly, from (44) if y, the solution of (38), is a polynomial of degree n, then $\delta_n \equiv 0$. Further, if $\delta_n = 0$, then $\delta_{n'} = 0$ for all $n' \geq n$. In the original paper on the Asymptotic Iteration Method (AIM) [16], the following theorem was proved.

Theorem 3 *Given λ_0 and s_0 in $C^\infty(a, b)$, the differential equation (38) has the general solution*

$$y(r) = \exp\left(-\int^r \alpha(t)dt\right)\left[C_2 + C_1\int^r \exp\left(\int^t (\lambda_0(\tau) + 2\alpha(\tau))d\tau\right)dt\right] \tag{45}$$

if for some $n > 0$

$$\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}} \equiv \alpha \qquad \text{or equivalently} \quad \delta_n \equiv s_n \cdot \lambda_{n-1} - s_{n-1} \cdot \lambda_n = 0. \tag{46}$$

The method can apply directly to the differential equation (9). We have considered here, as an example of the effectiveness of the method, the following two decatic potentials (see figure 3)

$$V(x) = 0.04x^{10} + 0.877x^8 + 5.5x^6 - 7.5x^4 + 2x^2, \tag{47}$$

and

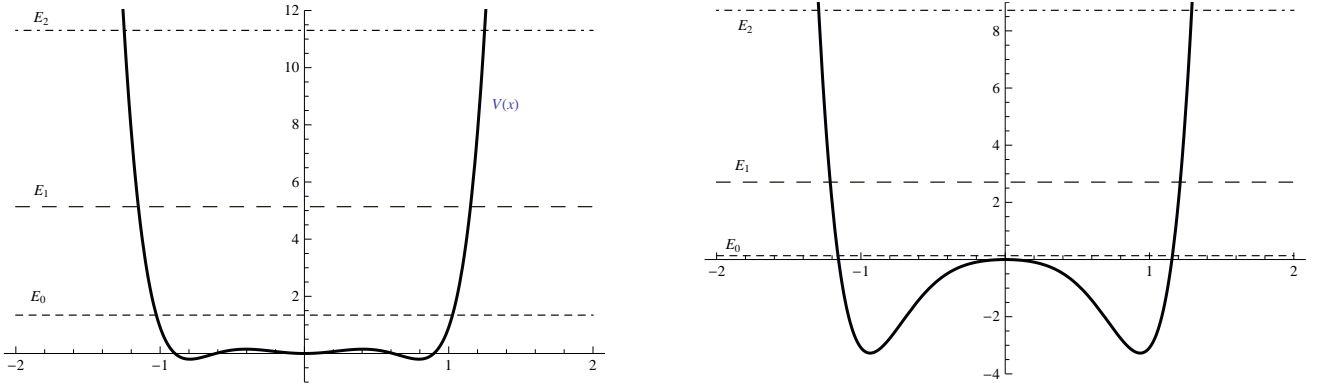$$V(x) = 0.04x^{10} + 0.877x^8 + 5.5x^6 - 7.5x^4 - 2x^2. \tag{48}$$

FIG. 3: The graph of the potentials $V(x) = 0.04x^{10} + 0.877x^8 + 5.5x^6 - 7.5x^4 + 2x^2$ and $V(x) = 0.04x^{10} + 0.877x^8 + 5.5x^6 - 7.5x^4 - 2x^2$, respectively.

For these specific values of the potential parameters (9), the AIM sequences λ_n and s_n, $n = 1, 2, \ldots$, can be initiated with

$$
\begin{cases}
\lambda_0(x) = 2\sqrt{a}x^5 + \frac{b}{\sqrt{a}}x^3 - \frac{(b^2 - 4ac)}{4a^{3/2}}x, \\
\\
s_0(x) = \frac{4ac - b^2}{8a^{3/2}} - E + \frac{64a^3e + 96a^{5/2}b - b^4 + 8ab^2c - 16a^2c^2}{64a^3}x^2 + \frac{8a^2d + 40a^{5/2} + b^3 - 4abc}{8a^2}x^4.
\end{cases}
\tag{49}
$$

and by means of the iteration sequences (37), we compute $\lambda_n(x)$ and $s_n(x)$, for $n = 1, 2, 3, \ldots$. The eigenvalues are then the roots of the termination condition (46), namely $\delta_n(x_0; E) = 0$. With several symbolic mathematical programs available (Maple; Mathematica, etc.), the computation of the roots of this equation, and thus the eigenvalues, by means of the iteration method, provided it is set up correctly, is a straightforward calculation, even for the higher iteration steps as shown in Table III. In principle, the computation of the roots of $\delta_n(x_0; E) = 0$ should be independent of the choice of $x = x_0$, nevertheless, the right choice of x_0, usually accelerates the rate of convergence to accurate eigenvalues within a reasonable number of iterations [14]. For our computation, we have fixed the initial value of x as $x_0 = 0$.

V. CONCLUSION

We have discussed exact and approximate solutions of Schrödinger's equations with even-power polynomial potentials. Our analysis of exact solutions is based on introducing necessary and sufficient conditions of the polynomial solution for a class of differential equations with polynomial coefficients. This analysis can serve as available tools to study more complicated physical systems implied by the differential equation. Although we confined our discussion in the present work to the decatic potential, our approach can be used to analyse different classes of oscillators with higher anharmonicities. For the arbitrary values of the potential parameters, we have applied the asymptotic iteration method to solve Schrödinger's equation for which the method proves to be extremely effective for finding the approximate solutions. We hope that the study presented here encourages further analysis of (both confined and unconfined) polynomial potentials.

n	$E_n(a=0.04, b=0.877, c=5.5, d=-7.5, e=2)$	n	$E_n(a=0.04, b=0.877, c=5.5, d=-7.5, e=-2)$
0	$1.342\ 322\ 779\ 564\ 646\ 216_{189}$	0	$0.135\ 429\ 448\ 914\ 739\ 200_{187}$
1	$5.136\ 482\ 243\ 202\ 476\ 766_{196}$	1	$2.708\ 255\ 201\ 038\ 304\ 023_{194}$
2	$11.304\ 205\ 821\ 188\ 349\ 432_{203}$	2	$8.719\ 471\ 783\ 294\ 745\ 896_{199}$
3	$19.577\ 677\ 092\ 617\ 956\ 963_{208}$	3	$16.684\ 057\ 789\ 758\ 348\ 655_{204}$
4	$29.438\ 055\ 052\ 333\ 889\ 749_{215}$	4	$26.234\ 813\ 143\ 437\ 435\ 201_{213}$
5	$40.723\ 083\ 272\ 298\ 272\ 621_{222}$	5	$37.249\ 467\ 272\ 037\ 347\ 758_{234}$

n	$E_n(a=0.01, b=0.1, c=1.0, d=3.250, e=12.5625)$	n	$E_n(a=0.01, b=0.1, c=1.0, d=3.050, e=11.5625)$
0	$3.750\ 000\ 000\ 000\ 000\ 000_6^{Exact}$	0	$3.611\ 850\ 704\ 712\ 528\ 538_{83}$
1	$11.653\ 048\ 680\ 350\ 115\ 469_{104}$	1	$11.250\ 000\ 000\ 000\ 000\ 000_6^{Exact}$
2	$20.358\ 948\ 672\ 177\ 131\ 878_{113}$	2	$19.713\ 587\ 721\ 031\ 462\ 373_{109}$
3	$29.850\ 701\ 419\ 298\ 223\ 478_{130}$	3	$28.983\ 552\ 585\ 573\ 622\ 185_{122}$
4	$40.098\ 623\ 827\ 649\ 000\ 672_{139}$	4	$39.026\ 595\ 688\ 356\ 661\ 175_{143}$
5	$51.071\ 414\ 767\ 832\ 192\ 856_{152}$	5	$49.808\ 260\ 402\ 594\ 522\ 700_{146}$

TABLE III: The first six energy eigenvalues (in units of $2m = \hbar = 1$) for the one-dimensional decatic potentials $V(x) = ax^{10} + bx^8 + cx^6 + dx^4 + ex^2$ for various values of parameters, calculated using AIM with $x_0 = 0$. The energies are accurate for the number of digits reported. The subscript refer to the number of iteration used by AIM.

Acknowledgements

Appendix

The first few polynomial solutions of the decatic power potentials using the recurrence relations (22) amd (26) are:

- For a zero-degree polynomial solution , $n = 0$ and $\chi_2(x) = 1$,

    1. $P_2(E) := 8a^{3/2}E + b^2 - 4ac = 0$,
    2. $96a^{5/2}b - b^4 + 8ab^2c - 16a^2c^2 + 64a^3e = 0$,
    3. $40a^{5/2} + b^3 - 4abc + 8a^2d = 0$.

- For a first-degree polynomial solution, $n = 1$ and $\chi_3(x) = c_0 + c_1 x$, we have $c_0 = 0$, and

    1. $P_3(E) := 8a^{3/2}E + 3(b^2 - 4ac) = 0$,
    2. $160a^{5/2}b - b^4 + 8ab^2c - 16a^2c^2 + 64a^3e = 0$,
    3. $56a^{5/2} + b^3 - 4abc + 8a^2d = 0$.

    Thus, if $c_1 = 1$,

$$\chi_3(x) = x.$$

- For a second-degree polynomial solution, $n = 2$ and $\chi_4(x) = c_0 + c_1 x + c_2 x^2$, we have $c_1 = 0$ and

    1. $P_4(E) := \left(8a^{3/2}E + 5(b^2 - 4ac)\right)P_2(E) + 2\left(96a^{5/2}b - b^4 + 8ab^2c - 16a^2c^2 + 64a^3e\right)P_0(E) = 0$,
    2. $\left(224a^{5/2}b - b^4 + 8ab^2c - 16a^2c^2 + 64a^3e\right)P_2(E) + 4096a^5 P_0(E) = 0$,
    3. $72a^{5/2} + b^3 - 4abc + 8a^2d = 0$.

    Further, with $c_1 = 0$, $c_2 = -(8a^{\frac{3}{2}}E + b^2 - 4ac)c_0/(16a^{\frac{3}{2}})$, and for $c_0 = 1$ we have

$$\chi_4(x) = P_0(E) - \frac{P_2(E)}{16a^{3/2}}x^2.$$

- For a third-degree polynomial solution, $n = 3$ and $\chi_5(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$, we have $c_0 = c_2 = 0$ and further

    1. $P_5(E) := \left(8a^{3/2}E + 5(b^2 - 4ac)\right)P_3(E) + 6\left(160a^{5/2}b - b^4 + 8ab^2c - 16a^2c^2 + 64a^3e\right)P_1(E) = 0$
    2. $\left(288a^{5/2}b - b^4 + 8ab^2c - 16a^2c^2 + 64a^3e\right)P_3(E) + 12288a^5 = 0$
    3. $88a^{5/2} + b^3 - 4abc + 8a^2d = 0$

  In addition, for $c_1 = 1$, $c_3 = -P_3(E)/(48a^{3/2})$, and the polynomial solution is

$$\chi_5(x) = x\left(P_1(E) - \frac{P_3(E)}{48a^{3/2}}x^2\right).$$

- For a fourth-degree polynomial solution, $n = 4$ and $\chi_6(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$, we have $c_1 = c_3 = 0$ and

    1. $P_6(E) := \left(8a^{3/2}E + 9(b^2 - 4ac)\right)P_4(E) + 12\left(224a^{5/2}b - b^4 + 8ab^2c - 16a^2c^2 + 64a^3e\right)P_2(E) +$
       $+ 1536a^{5/2}\left(40a^{5/2} + b^3 - 4abc + 8a^2d\right)P_0(E) = 0$
    2. $\left(352a^{5/2}b - b^4 + 8ab^2c - 16a^2c^2 + 64a^3e\right)P_4(E) + 24576a^5 P_2(E) = 0$
    3. $104a^{5/2} + b^3 - 4abc + 8a^2d = 0$

  In addition, with $c_0 = 1$, $c_2 = -\dfrac{P_2(E)}{16a^{3/2}}$ and $c_4 = \dfrac{P_4(E)}{1536a^3}$, and the polynomial solution is

$$\chi_6(x) = P_0(E) - \frac{P_2(E)}{16a^{3/2}}x^2 + \frac{P_4(E)}{1536a^3}x^4.$$

---

[1] V. S. Adamchik and A. D. Lizarev, *Wronskians of solutions of a class of differential equations with polynomial coefficients*, Ukrainian Mathematical Journal **40** (1988) 583-587.

[2] R. Adhikari, R. Dutt and Y. P. Varshni, *Exact solutions for polynomial potentials using supersymmetry inspired factorization method*, Phys. Lett. A **141** (1989) 18.

[3] D. Agboola and Y.-Z. Zhang, *Unified derivation of exact solutions for a class of quasi-exactly solvable models*, J. Math. Phys. **53** (2012) 042101.

[4] D. Agboola and Y.-Z. Zhang, *Exact solutions of the Schrödinger equation with spherically symmetric octic potential*, Mod. Phys. Lett. A **27** (2012) 1250112 .

[5] D. Agboola and Y.-Z. Zhang, *Quasi-exactly solvable relativistic soft-core Coulomb models*, Annals of Physics **327** (2012) 2275-2287.

[6] A. D. Alhaidari, *Representation reduction and solution space contraction in quasi-exactly solvable systems*, J. Phys. A: Math. Theor. **40** (2007) 6305 - 6328.

[7] H. Azad, A. Laradji and M. T. Mustafa, *Polynomial solutions of differential equations*, Advances in Difference Equations **2011** (2011) 58.

[8] T. Barakat, *The asymptotic iteration method for the eigenenergies of the anharmonic oscillator potential $V(x) = Ax^{2\alpha} + Bx^2$*, Phys. Letters A **344** (2005) 411 - 417.

[9] M. Bansal, S. Srivastava, and Vishwamittar, *Energy eigenvalues of double-well oscillator with mixed quartic and sextic anharmonicities*, Phys. Rev. A **44** (1991) 8012 - 8019.

[10] C. M. Bender and G. V. Dunne, *Quasi-exactly solvable systems and orthogonal polynomials*, J. Math. Phys. **37**, 6 (1997).

[11] C. M. Bender and M. Monou, *New quasi-exactly solvable sextic polynomial potentials*, J. Phys. A: Math. Gen. **38** (2005) 2179.

[12] A. Boumenir, *The recovery of even polynomial potentials*, Applied Mathematics and Computation, **215** (2009) 2914 - 2926.

[13] A. S. de Castro and A. de Souza Dutra, *Approximate analytical states of a polynomial potential: an example of symmetry restoration*, Phys. Letters A **269** (2000) 281 - 286.

[14] B. Champion, R. L. Hall, and N. Saad, *Asymptotic iteration method for singular potentials*, Int. J. Mod. Phys. A **23** (2008) 1405 - 1415.

[15] R. N. Chaudhuri and M. Mondal, *Improved Hill determinant method: General approach to the solution of quantum anharmonic oscillators*, Phys. Rev. A **43** (1991) 3241 - 3246.

[16] H. Ciftci, R. L. Hall and N. Saad, *Asymptotic iteration method for eigenvalue problems*, J. Phys. A **36** (2003) 11807-11816.

[17] H. Ciftci, R. L. Hall, N. Saad, and E. Dogu, *Physical applications of second-order linear differential equations that admit polynomial solutions*, J. Phys. A: Math. Theor. 43 (2010) 415206.

[18] J. Dziubański, *A note on Schrödinger operators with polynomial potentials* Colloquium Mathematicum **78** (1998) 149 - 161.

[19] Shi-Hai Dong, *Quantum monodromy in the spectrum of Schrödinger equation with a decatic potential*, Int. J. Theor. Phys. **41** (2002) 89- 99.

[20] Shi-Hai Dong, *The Ansatz method for analyzing Schrödinger's equation with three anharmonic potentials in D dimensions*, Found. Phys. Lett. **15** (2002) 385 - 395.

[21] Shi-Hai Dong, *On the solutions of the Schrödinger equation with some anharmonic potentials: Wave function ansatz,* Phys. Scr. **65** (2002) 289 - 295.

[22] A. Eremenko, A. Gabrielov and B. Shapiro, *High energy eigenfunctions of one-dimensional Schrödinger operators with polynomial potentials*, Comput. Methods Funct. Theory **8** (2008) 513529.

[23] F. M. Fernández, *Bound-state eigenvalues for polynomial potentials*, Phys. Rev. A **44** (1991) 33363339.

[24] F. M. Fernández and R. Guardiola, *Accurate eigenvalues and eigenfunctions for quantum-mechanical anharmonic oscillators,* J. Phys. A Math. Gen. **26** (1993) 7169-7180.

[25] G. P. Flessas, *Exact solutions for anharmonic oscillators,* J. Phys. A: Math. Gen. **14** (1981) L209 - L211.

[26] A. de Freitas, P. Martín, E. Castro, J. L. Paz, *Eigenvalues and eigenfunctions for the ground state of polynomial potentials*, Phys. Lett. A **362** (2007) 371-376.

[27] F. Finkel, A. González-López, N. Kamran, Peter J. Olver, Miguel A. Rodríguez, *Lie algebras of differential operators and partial integrability* in: Proceedings of IV Workshop on Differential Geometry and its Applications; Santiago de Compostela, Spain, 1995.

[28] F. J. Gómez and J. Sesma, *Quantum anharmonic oscillators: a new approach*, J. Phys. A: Math. Gen. **38** (2005) 3193 - 3202.

[29] A. González-López, N. Kamran, and P. J. Olver, *Real Lie algebras of differential operators, and quasi-exactly solvable potentials*, Phil. Trans. Roy. Soc. London A **354** (1996), 1165-1193.

[30] K. Gustav and R. Colditz, *Potential adaption: Use of polynomials as anharmonic oscillator potentials,* Int. J. Quant. Chem. **21** (1982) 1091 - 1095.

[31] R. L. Hall and N. Saad, *Bounds on Schrödinger eigenvalues for polynomial potentials in N-dimensions*, J. Math. Phys. **38** (1997) 4909.

[32] R. L. Hall, N. Saad and K. D. Sen, *Discrete spectra for confined and unconfined $-a/r + br^2$ Potentials in d-Dimensions*, J. Math. Phys. **52**, 092103 (2011).

[33] R. L. Hall, N. Saad and K. D. Sen, *Spectral characteristics for a spherically confined $-a/r + br^2$ potential*, J. Phys. A: Math. Theor. **44**, 185307 (2011).

[34] R. L. Hall, N. Saad and K. D. Sen, *Soft-core Coulomb potential and Heun's differential equation*, J. Math. Phys. **51**, 022107 (2010).

[35] R. L. Hall, N. Saad, K. D. Sen and H. Ciftci, *Energies and wave functions for a soft-core Coulomb potential* Phys. Rev. A **80**, 032507 (2009).

[36] I. A. Ivanov, *Sextic and octic anharmonic oscillators: connection between strong-coupling and weak-coupling expansions*, J. Phys. A: Math. Gen. 31 (1998) 5697 - 5704.

[37] M. Jafarpour and D. Afshar, *Calculation of energy eigenvalues for the quantum anharmonic oscillator with a polynomial potential*, J. Phys. A: Math. Gen. **35** (2002) 87 - 92.

[38] M. Jafarpour and D. Afshar, *Energy levels for the pure λx^{2m} potentials*, Journal of Sciences, Islamic Republic of Iran **18** (2007) 75 - 81.

[39] E. G. Kalnins, W. D. Miller and G. S. Pogosyan, *Exact and quasi-exact solvability of second-order superintegrable quantum systems: I. Euclidean space preliminaries,* J. Math. Phys. **47** (2006) 033502.

[40] N. Kamran and P. J. Olver, *Lie algebras of differential operators and Lie-algebraic potentials*, J. Math. Anal. Appl. **145** (1990), 342-356.

[41] Q. D. Katatbeh1, R. L. Hall and N. Saad, *Eigenvalue bounds for polynomial central potentials in d-dimensions,* J. Phys. A: Math. Theor. **40** (2007) 13431.

[42] P Kościk and A. Okopińska, *Application of the Fröbenius method to the Schrödinger equation for a spherically symmetric potential: an anharmonic oscillator*, J. Phys. A: Math. Gen. **38** (2005) 7743-7755.

[43] V. E. Kruglov, *Solution of a second-order linear differential equation with polynomial coefficients and Fuchsian point at zero*, Differential Equations **47** (2011) 20-28.

[44] E. Z. Liverts and V. B. Mandelzweig, *Approximate analytic solutions of the Schrödinger equation for the generalized anharmonic oscillator*, Phys. Scr. **77** (2008) 025003.

[45] E. Magyari, *Exact quantum-mechanical solutions for anharmonic oscillators*, Phys. Letters A **81** (1981) 116 - 118.

[46] Y. Meurice, *Arbitrarily accurate eigenvalues for one-dimensional polynomial potentials*, J. Phys. A: Math. Gen. **35** (2002) 8831.

[47] A. Nanayakkara and I. Dasanayake, *Analytic semiclassical energy expansions of general polynomial potentials*, Phys. letters A **294** (2002) 158-162.

[48] V. V. Napalkov and A. M. Dilmukhametova, *On one class of differential equations with polynomial coefficients* Doklady Mathematics **79** (2009) 97-99.

[49] F. Pan, J. R. Klauder and J. P. Draayer, *Quasi-exactly solvable cases of an N-dimensional symmetric decatic anharmonic oscillator*, Phys. Letts. A **262**(1999) 131.

[50] E. D. Rainville, *Necessary conditions for polynomial solutions of certain Riccati equations* Amer. Math. Monthly **43** (1936) 473.

[51] R. Roychoudhury, P. Roy, M. Znojil, and G. Lévai, *Comprehensive analysis of conditionally exactly solvable models*, J. Math. Phys. **42** (2001) 1996.

[52] N. Saad, R. L. Hall and H. Ciftci, *Sextic anharmonic oscillators and orthogonal polynomials*, J. Phys. A: Math. Gen. **39** (2006) 8477-8486.

[53] N. Saad, R. L. Hall and H. Ciftci, *Criterion for polynomial solutions to a class of linear differential equation of second order*, J. Phys. A: Math. Gen. **39** (2006) 13445-13454.

[54] H. Taşeli, *The influence of the boundedness of polynomial potentials on the spectrum of the Schrdinger equation* J. Comput. Phys. **101** (1992) 252 - 255.

[55] H. Tezuka, *Confinement by polynomial potentials*, Z. Phys. C **65** (1995) 101-104.

[56] A. Turbiner, *On polynomial solutions of differential equations*, J. Math. Phys. **33** (1992) 3989-3993.

[57] A. Turbiner, *Quasi-exactly-solvable problems and sl(2) algebra*, Comm. Math. Phys. **118** (1988) 467 - 474.

[58] A. G. Ushveridze, *Quasi-exactly solvable models in quantum mechanics,* Bristol: Institute of Physics Publishing (1994).

[59] Mamta Vishwarmittar, *Energy eigenvalues for anharmonic and double-well oscillators with even power polynomial potential*, Physica A **216** (1995) 452 - 458.

[60] A. Voros, *Exact resolution method for general 1D polynomial Schrödinger equation,* J. Phys. A: Math. Gen. **32** (1999) 5993.

[61] H. J. Weber, *Connections between Romanovski and other polynomials*, Central European Journal of Mathematics **5** 415 (2007).

[62] E. J. Weniger, *Construction of the strong coupling expansion for the ground state energy of the quartic, sextic, and octic anharmonic oscillator via a renormalized strong coupling expansion*, Phys. Rev. lett. **77** (1996) 2859 - 2862.

[63] M. Znojil, *Polynomial potentials and a hidden symmetry of the Hill-determinant eigenvalue method*, Phys. Letters A **116** (1986) 207 - 209.

[64] M. Znojil, *Re-construction of polynomial potentials with a perturbation-interpolation constraint*, Quantum Inversion Theory and Applications: Proceedings of the 109th W.E. Heraeus, Ed. H. V. Geramb, Lecture Notes in Physics, vol. 427,(1993) 458 - 464.

[65] M. Znojil, *Perturbation method for non-square Hamiltonians and its application to polynomial oscillators*, Phys. Letters A **341** (2005) 67 - 80.

[66] M. Znojil and D. Yanovich, *New type of exact solvability and of a hidden nonlinear dynamical symmetry in anharmonic oscillators,* Proceedings of Institute of Mathematics of NAS of Ukraine **50** (2004) 1010-1017.

[67] Yao-Zhong Zhang *Exact polynomial solutions of second order differential equations and their applications*, J. Phys. A: Math. Theor. **45** 065206 (2012).